4 – Structural Design Formulae

Short Term Loads

If a plastic part is subjected to a load for only a short time (10-20 minutes) and the part is not stressed beyond its elastic limit, then classical design formulae found in engineering texts as reprinted here can be used with sufficient accuracy. These formulae are based on Hooke's Law which states that in the elastic region the part will recover to its original shape after stressing, and that stress is proportional to strain.

Tensile Stress – Short Term

Hooke's law is expressed as:

$$\varepsilon = \frac{\sigma}{E}$$

where:

ε = strain (%/100) = $\frac{\Delta \ell}{\ell}$

σ = stress (MPa), defined as $\sigma = \frac{F}{A}$

E = modulus of elasticity (MPa)

F = total force (N)

A = total area (mm^2)

ℓ = length of member (mm)

$\Delta \ell$ = elongation (mm)

Bending Stress

In bending, the maximum stress is calculated from:

$$\sigma_b = \frac{My}{I} = \frac{M}{Z}$$

where:

σ_b = bending stress (MPa)

M = bending moment (Nmm)

I = moment of inertia (mm^4)

y = distance from neutral axis to extreme outer fibre (mm)

$Z = \frac{I}{y}$ = section modulus (mm^3)

The I and y values for some typical cross-sections are shown in Table 4.01.

Beams

Various beam loading conditions are shown in Table 4.02.

Beams in Torsion

When a plastic part is subjected to a twisting moment, it is considered to have failed when the shear strength of the part is exceeded.

The basic formula for torsional stress is: $\tau = \frac{M_T r}{K}$

where:

τ = Shear stress (MPa)

M_T = Twisting Moment (N · mm)

r = Distance to centre of rotation (mm)

K = Torsional Constant (mm^4)

Formulae for sections in torsion are given in Table 4.03.

To determine Θ, angle of twist of the part whose length is ℓ, the equation shown below is used:

$$\Theta = \frac{M_T \ell}{KG}$$

where:

Θ = angle of twist (radians)

K = Torsional Constant (mm^4)

ℓ = length of member (mm)

G = modulus in shear (MPa)

To approximate G, the shear modulus, use the equation,

$$G = \frac{E}{2\,(1+\nu)} \quad \text{(for isotropic materials)}$$

where:

E = Modulus (MPa)

ν = Poisson's Ratio; generally for plastics:

$E < 500$: $\nu = 0{,}45$

$500 < E < 2500$: $\nu = 0{,}40$

$E > 2500$: $\nu = 0{,}35$

Tubing and Pressure Vessels

Internal pressure in a tube, pipe or pressure vessel creates three (3) types of stresses in the part: Hoop, meridional and radial. See Table 4.04.

Buckling of Columns, Rings and Arches

The stress level of a short column in compression is calculated from the equation,

$$\sigma_c = \frac{F}{A}$$

The mode of failure in short columns is compressive failure by crushing. As the length of the column increases, however, this simple equation becomes invalid as the column approaches a buckling mode of failure. To determine if buckling will be a factor, consider a thin column of length ℓ, having frictionless rounded ends and loaded by force F. As F increases, the column will shorten in accordance with Hooke's Law. F can be increased until a critical value of F_C is reached.

Any load above F_C will cause the column to buckle.

In equation form:

$$F_C = \frac{\pi^2 E_t I}{\ell^2}$$

In this formula, which is called the Euler Formula for round ended columns:

E_t = Tangent modulus at stress σ_C
I = moment of inertia of cross section.

A safety factor of 3 to 4 should be applied.

Thus, if the value for F_C is less than the allowable load under pure compression, the buckling formula should be used.

If the end conditions are altered from the round ends, as is the case with most plastic parts, then the critical load is also altered. See Table 4.05 for additional end effect conditions for columns.

Flat Plates

Flat plates are another standard shape found in plastic part design. Their analysis can be useful in the design of such products as pump housings and valves.

A few of the most commonly used geometrics are shown in Table 4.06.

Arbitrary Structures

A lot of injection moulded parts have a shape which cannot be compared with one of the structures from Tables 4.01 to 4.06.

Deformations of, and stresses in these parts, can be analysed by using the Finite Element method.

For recommended material properties, mesh to be used, simulation of loads and boundary conditions, and assessment of results, DuPont's Engineering Polymers Technical Service can provide assistance.

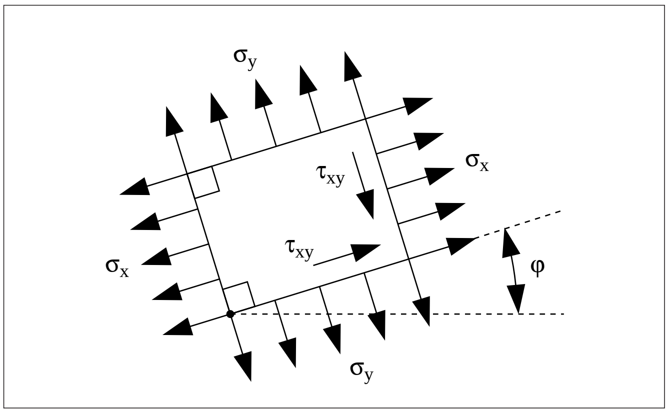
Equivalent Stress / Allowable Stress

Tensile and bending stresses are always pependicular (normal) to a considered cross section, while shear stresses act in the cross-sectional plane. At a given location there are often multiple stress components acting at the same time. To express the "danger" of such a multiaxial stress state by only one number, "equivalent stresses" are used. A widely known formula to calculate the equivalent stress in isotropic materials is the "Von Mises" criterium (two-dimensional):

$$\sigma_{eq,\ VonMises} = \sqrt{\sigma_x{}^2 + \sigma_y{}^2 - \sigma_x\, \sigma_y + 3\tau_{xy}}$$

with: σ_x, σ_y: normal stress
τ_{xy}: shear stress

according:



Another well known criterium is that of "Tresca":

$$\sigma_{eq,\ Tresca} = \sigma_1 - \sigma_2$$

with: σ_1 = maximum principal stress
σ_2 = minimum principal stress (≤ 0)

Principal stresses are normal stresses at a given location, whereby the cross-sectional plane is rotated in such a way that the shear stress $\tau_{xy} = 0$, see Figure above.

The equivalent stress should be less than the yield strength at design conditions, as measured on test specimen; whereby application dependant safety factors must be considered.

$$\sigma_{eq} \leq \sigma_{all} = \sigma_{yield}/S$$

with: S = Safety factor (≥ 1).
Suggested for static loads: S = 1,5–2,0.

Brittle Materials

For brittle materials ($\epsilon_B < 5\%$) also the following conditions should be satisfied:

$$\sigma_{eq} \leq \frac{\epsilon_B E}{S \times SCF}$$

with: ϵ_B = elongation at break (%/100)
E = modulus of elasticity
S = safety factor (≥ 1)
SCF = stress concentration factor (≥ 1):
normal design = SCF = 3,0
nicely filleted = SCF = 2,0
sharp corners = SCF = 4,0 – 6,0

Modulus for Isotropic Materials

For the analyses of deformations, stresses and allowable loads, the modulus of elasticity is required input. Values for most DuPont engineering polymers can be found in CAMPUS, which database can be down-loaded for free via Internet. One should however not forget that the values given in CAMPUS are measured according ISO standards, which standards may not be comparable with the practical situation, for instance with respect to applied load, load duration and orientation of glass fibres in case of glass-fibre reinforced materials.

The following guidelines should lead to more precise results when analyses are carried out with isotropic material properties:

– *Static analysis,*
  use stress-strain curve at design temperature,
  – non reinforced materials:
    use appearant modulus at 1% strain.
  – reinforced materials:
    define appearant modulus at 0,5% strain,
    use 90% of appearant modulus for highly oriented fibres;
    use 80% of appearant modulus for good oriented fibres;
    use 50% of appearant modulus for poorly oriented fibres.

  The appearant modulus is defined by the slope of the line, connecting the origin off the stress-strain curve with a point at a given strain:

  $E_{app} = \sigma_0/\varepsilon_0$, see also Fig 4.01.

  For polyamides the stress-strain curves at 50 RH (conditioned) should be selected.

  Corrections for *creep* are applicable for load durations longer than 0,5 hour, see also paragraph "Long Term Loads". Then, instead of the standard stress-strain curve, an isochronous stress strain-curve at the design temperature and for the applicable time should be used.

– *Dynamic analysis,*
  use Dynamic Mechanical Analyser measurements,
  – non reinforced materials:
    use value at design temperature.
  – reinforced materials:
    use 85 % of value at design temperature.

Orthotropic Materials

Glass fibre reinforced plastics have properties (modulus of elasticity, coefficient of linear thermal expansion, tensile strength), which are significantly different for in-flow and transverse to flow directions. Analyses with orthotropic (anisotropic) materials is in general only possible with the finite element method. In this approach, a flow analysis is included to calculate the material orientations of the elements. Formulae to calculate the equivalent stresses in othotropic materials exist, but are complicated. A more simple (but still good enough) approach is to adjust the allowable stress ($\sigma_{tensile}/S$), to a value applicable for the given orientation.

Other Loads

Fatigue Resistance

When materials are stressed cyclically they tend to fail at levels of stress below their ultimate tensile strength. The phenomenon is termed ''fatigue failure''.

Fatigue resistance data (in air) for injection moulded material samples are shown in the product modules. These data were obtained by stressing the samples at a constant level at 1800 cpm and observing the number of cycles to failure at each testing load on a Sonntag-Universal testing machine.

Experiment has shown that the frequency of loading has no effect on the number of cycles to failure at a given level of stress, below frequencies of 1800 cpm. However, it is probable that at higher frequencies internal generation of heat within the specimen may cause more rapid failure.

Impact Resistance

End-use applications of materials can be divided into two categories.

– Applications where the part must withstand impact loadings on only a few occasions during its life.
– Applications where the part must withstand repeated impact loadings throughout its life.

Materials considered to have good impact strength vary widely in their ability to withstand repeated impact. Where an application subject to repeated impact is involved, the designer should seek specific data before making a material selection. Such data can be found in the product modules for DELRIN® resin and ZYTEL® resin, both of which demonstrate excellent resistance to repeated impact.

The energy of an impact must either be absorbed or transmitted by a part, otherwise mechanical failure will occur. Two approaches can be used to increase the impact resistance of a part by design:

– Increase the area of load application to reduce stress level.
– Dissipate shock energy by designing the part to deflect under load.

Designing flexibility into the part significantly increases the volume over which impact energy is absorbed. Thus the internal forces required to resist the impact are greatly reduced.

It should be emphasized that structural design for impact loading is usually a very complex and often empirical exercice. Since there are specific formulations of engineering materials available for impact applications, the designer should work around the properties of these materials during the initial drawing stage, and make a final selection via parts from a prototype tool which have been rigorously tested under actual end-use conditions.

Thermal Expansion and Stress

The effects of thermal expansion should not be overlooked in designing with thermoplastics.

For unreinforced plastic materials, the thermal expansion coefficient may be six to eight times higher than the coefficient of most metals. This differential must be taken into account when the plastic part is to function in conjunction with a metal part. It need not be a problem if proper allowances are made for clearances, fits, etc.

For example, if a uniform straight bar is subjected to a temperature change ΔT, and the ends are not constrained, the change in length can be calculated from:

$$\Delta L = \Delta T \times \alpha \times L$$

where:

ΔL	= change in length	(mm)
ΔT	= change in temperature	(°C)
α	= thermal expansion coefficient	(mm/mm°C)
L	= original length	(mm)

If the ends are constrained, the stress developed is:

$$\sigma = \Delta T \times \alpha \times E$$

where:

σ	= compressive stress	(MPa)
E	= modulus	(MPa)

The thermal stresses in a plate, constrained in two directions are:

$$\sigma = \Delta T \times \alpha \times E / (1 - \nu)$$

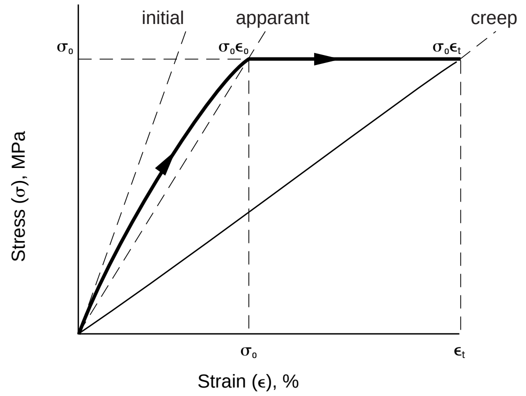
where: ν = Poissons ratio

When a plastic part is constrained by metal, the effect of stress relaxation as the temperature varies must be considered, since the stiffer metal part will prevent the plastic part from expanding or contracting, as the case may be.

Long Term Loads

Plastic materials under load will undergo an initial deformation the instant the load is applied and will continue to deform at a slower rate with continued application of the load. This additional deformation with time is called ‘‘creep’’.

Creep, defined as strain (%) over a period of time under constant stress, can occur in tension, compression, flexure or shear. It is shown on a typical stress-strain curve in Fig. 4.01.

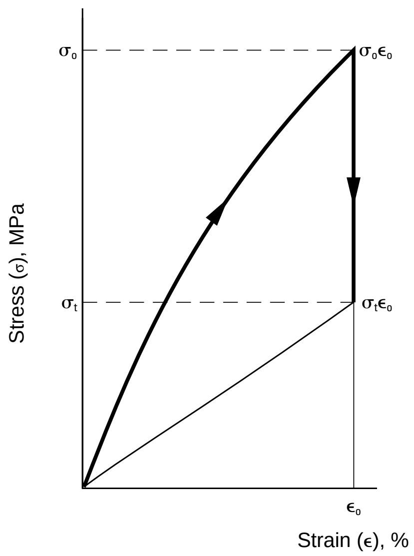


Creep between time t and $t_o = \epsilon_t - \epsilon_o$%. The creep modulus E_c for design in creep applications at stress σ_o and time t is the slope of the secant from the origin to the point ($\sigma_o\,\epsilon_t$).

Fig. 4.01 **Creep**

The stress required to deform a plastic material a fixed amount will decay with time due to the same creep phenomenon. This decay in stress with time is called stress relaxation.

Stress relaxation is defined as the decrease, over a given time period, of the stress (MPa) required to maintain constant strain. Like creep, it can occur in tension, compression, flexure or shear. On a typical stress-strain curve it is shown in Fig. 4.02.



Relaxation between time t and $t_o = \sigma_o - \sigma_t$. The relaxation modulus E_R for design in relaxation applications (e.g., press fits) at time T is the slope of the secant from the origin to the point ($\sigma_t\,\epsilon_o$).

Fig. 4.02 **Relaxation**

Laboratory experiments with injection moulded specimens have shown that for stresses below about ⅓ of the ultimate tensile strength of the material at any temperature, the secant moduli in creep and relaxation at any time of loading may be considered similar for engineering purposes. Furthermore, under these conditions, the secant moduli in creep and relaxation in tension, compression and flexure are approximately equal.

A typical problem using creep data found in the properties sections is shown below:

Cylinder under Pressure

Example 1: A Pressure Vessel Under Long Term Loading

As previously noted, it is essential for the designer to itemise the end-use requirements and environment of a part before attempting to determine its geometry. This is particularly true of a pressure vessel, where safety is such a critical factor. In this example, we will determine the side wall thickness of a gas container which must meet these requirements:

a) retain pressure of 0,7 MPa;

b) for 10 years;

c) at 65°C.

The inside radius of the cylinder is 9 mm and the length is 50 mm. Because the part will be under pressure for a long period of time, one cannot safely use short-term stress-strain data but should refer to creep data or, preferably, longterm burst data from actual pressure cylinder tests. Data typical of this sort for 66 nylons is shown in Fig. 4.03 which plots hoop stress versus time to failure for various moisture contents at 65°C.

Actually, ZYTEL® 101 would be a good candidate for this application as it has high impact strength in the 50% RH stabilized condition and the highest yield strength of unreinforced nylons.

Referring to the curve, we find a hoop stress level of 19 MPa at 10 years, and this can be used as the design stress. The hoop stress formula for a pressure vessel is:

$$t = \frac{Pr}{\sigma} \times F.S.$$

where:

t = wall thickness, mm
P = internal pressure, MPa
r = inside diameter, mm
σ = design hoop stress, MPa
F.S. = factor of safety = 3 (example)

$$t = \frac{(0{,}7)\ (9)\ (3)}{19} = 1{,}0 \text{ mm}$$

The best shape to use for the ends of the cylinder is a hemisphere. Hemispherical ends present a design problem if the cylinder is to stand upright. A flat end is unsatisfactory, as it would buckle or rupture over a period of time. The best solution, therefore, is to mould a hemispherical end with an extension of the cylinder or skirt to provide stability (Fig. 4.04).

For plastic parts under long term loads, stresses, deflections, etc. are calculated using classical engineering formula with data from the creep curves. The Elastic or Flexural Modulus is not used but rather the Creep Modulus, in equation form:

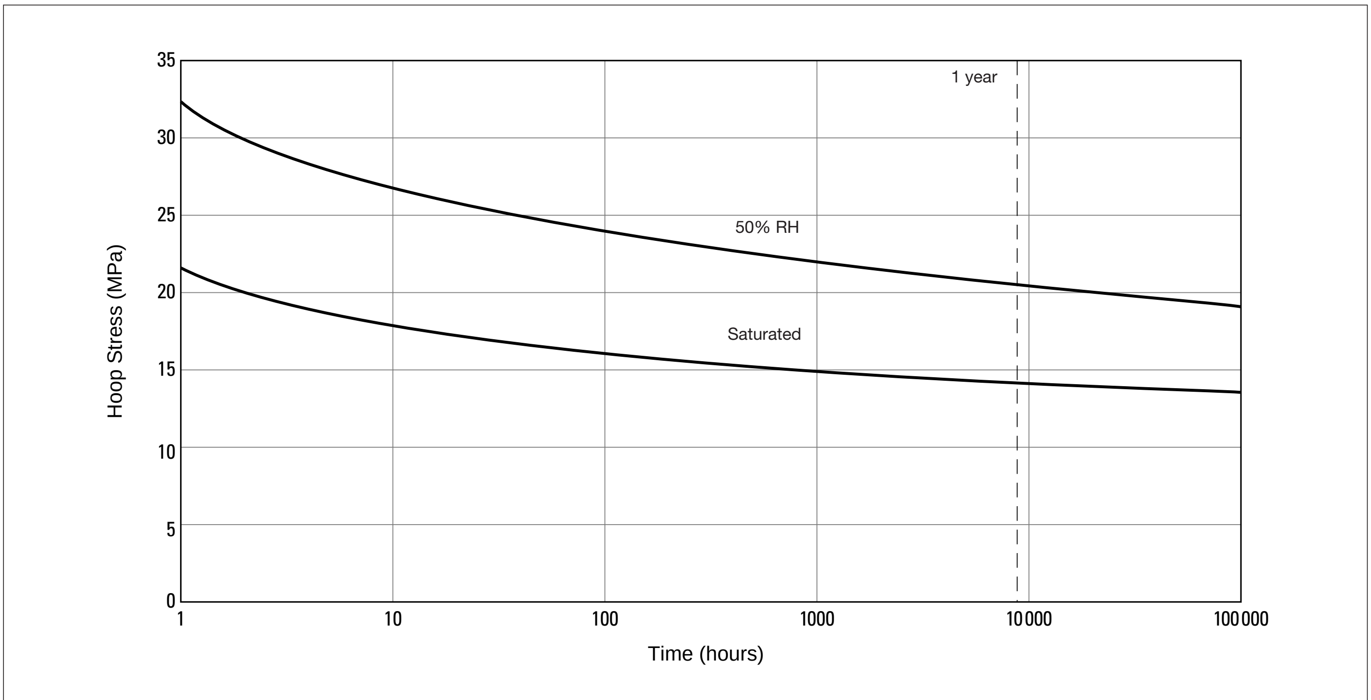


Fig. 4.03 **Hoop stress vs. time to failure, ZYTEL® 101 at 50% RH and at saturation 65°C**

$$E_c = \frac{\sigma}{\epsilon_o + \epsilon_c}$$

σ = stress under consideration (MPa)

ϵ_o = initial strain (%/100)

ϵ_c = creep strain (%/100)

For the strains ε in the above equation, there often can be written:

$$\epsilon_o + \epsilon_c = \frac{\sigma}{E_o} + \frac{\sigma}{E_o} At^B = \frac{\sigma}{E_o}(1 + At^B)$$

where:

E_o = apparant modulus at design conditions (MPa)
t = time (h)
A, B = material constants

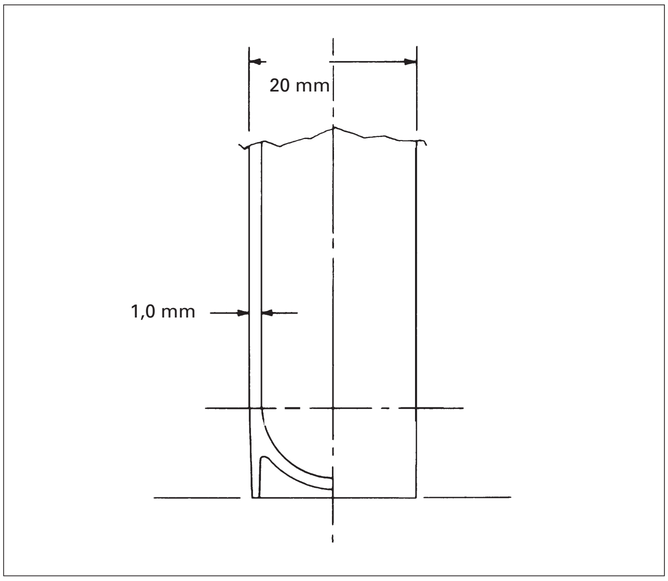


Fig. 4.04 **Design for a pressure vessel under long terme loading**

Tensile Loads

Long Term – Examples

Determine the stress and elongation of the tubular part shown in Fig. 4.05 after 1000 hours.

Material = ZYTEL® 101, 23°C, 50% RH
Tensile Loading = 1350 N
Outside Diameter = 25 mm
Wall Thickness = 1,3 mm
Length = 152 mm

$$\text{Stress} = \frac{F}{A} =$$

$$\frac{4\,F}{\pi\,(Do^2 - Di^2)} = \frac{(4)\,(1350)}{\pi\,(25^2 - 22{,}4^2)} = 14\ \text{MPa}$$

From Fig. 4.06 at 14 MPa and 1000 hours, the strain is 3%. Therefore, the elongation equals:

$L \times \Delta L = 152 \times 0{,}03 = 4{,}57$ mm.

(In this example there was assumed, that the creep in tension is equal to creep in flexure, which is not always correct.)

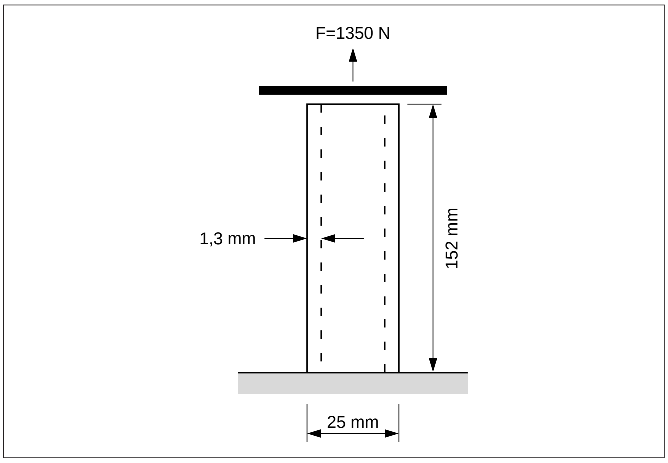


Fig. 4.05 **Example of creep in tubular part**

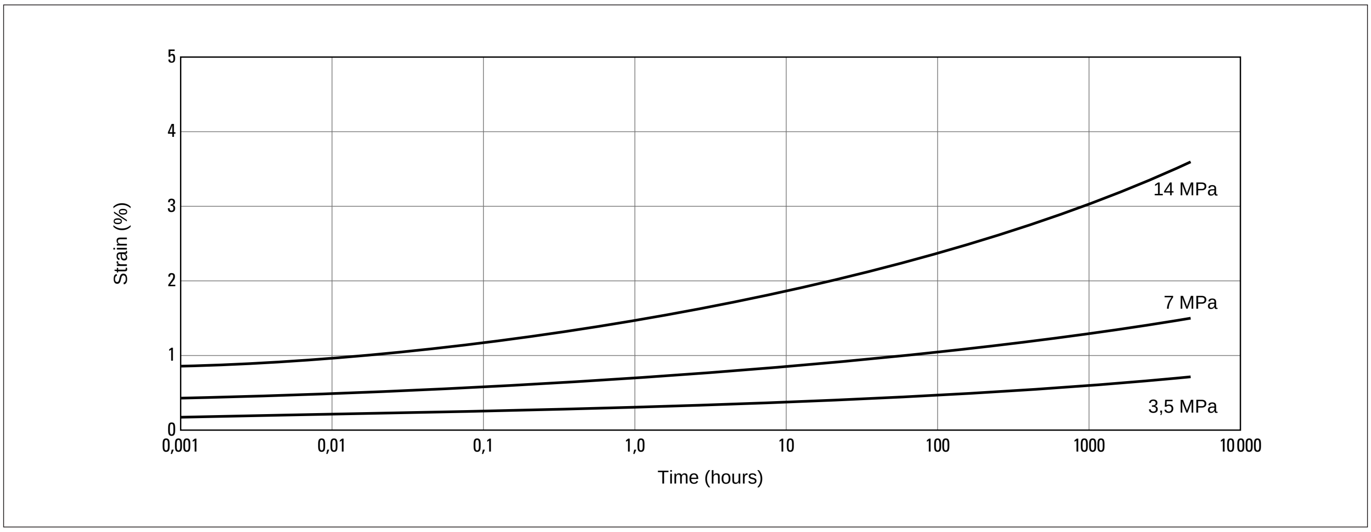


Fig. 4.06 **Creep in flexure of ZYTEL® 101, 23°C, 50% RH; $\epsilon_t = \sigma\,(1 + 0{,}65\,t^{0.2}) / E_o$; ($E_o$ = 1550 MPa)**

Ribs and Strengthening Members

Ribs can be used to augment greatly the section stiffness of simple beams. Often, thick sections can be replaced by sections of smaller cross-sectional area (such as ‘‘T’’ beams) with significant savings in material. However, checks should be made to ensure that acceptable design stress levels for the material are observed.

The designer must take great care in using ribs in a moulded part. Where they may provide the desired stiffness, it is also possible that the ribbing will distort the part after moulding. Therefore, they should be specified with caution as it is easier and cheaper to add ribs to a mould than it is to remove them.

Ribs and strengthening members should be ½ – ⅔ as thick as the walls they reinforce and deep ribs may require ¼ – ½° of taper for easy ejection from the mould (see Table 3.01). The reasons for using a thinner wall for the ribs are two: to minimize sink marks in the exterior surface caused by increased shrinkage at the intersection of rib and wall; and to prevent part distortion which again could be caused by the heavier section of the intersection. Figure 4.07 illustrates this effect.

By drawing a circle at the intersection of the rib and wall, a means is obtained to compare section thickness. A rib thickness (T) equal to the wall thickness, combined with a radius of 0,5 T, produces a circle with a diameter of 1,5 T or 50 per cent greater thant the wall thickness. Increasing the radius beyond 0,5 T would not significantly strengthen the corners, but would enlarge the inscribed circle, making the possibility of having voids in this area greater

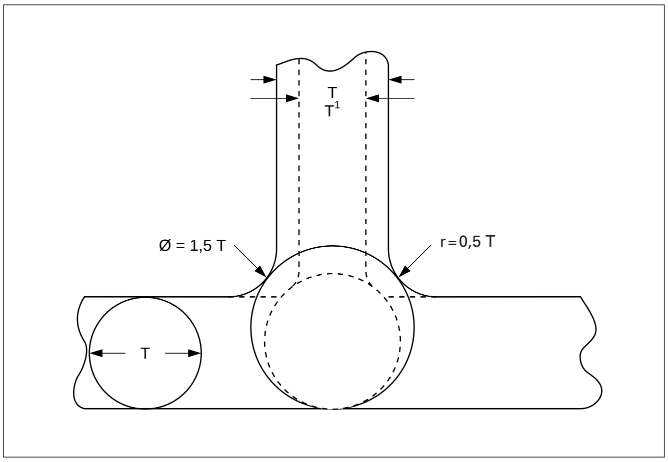


Fig. 4.07 **Rib dimensions**

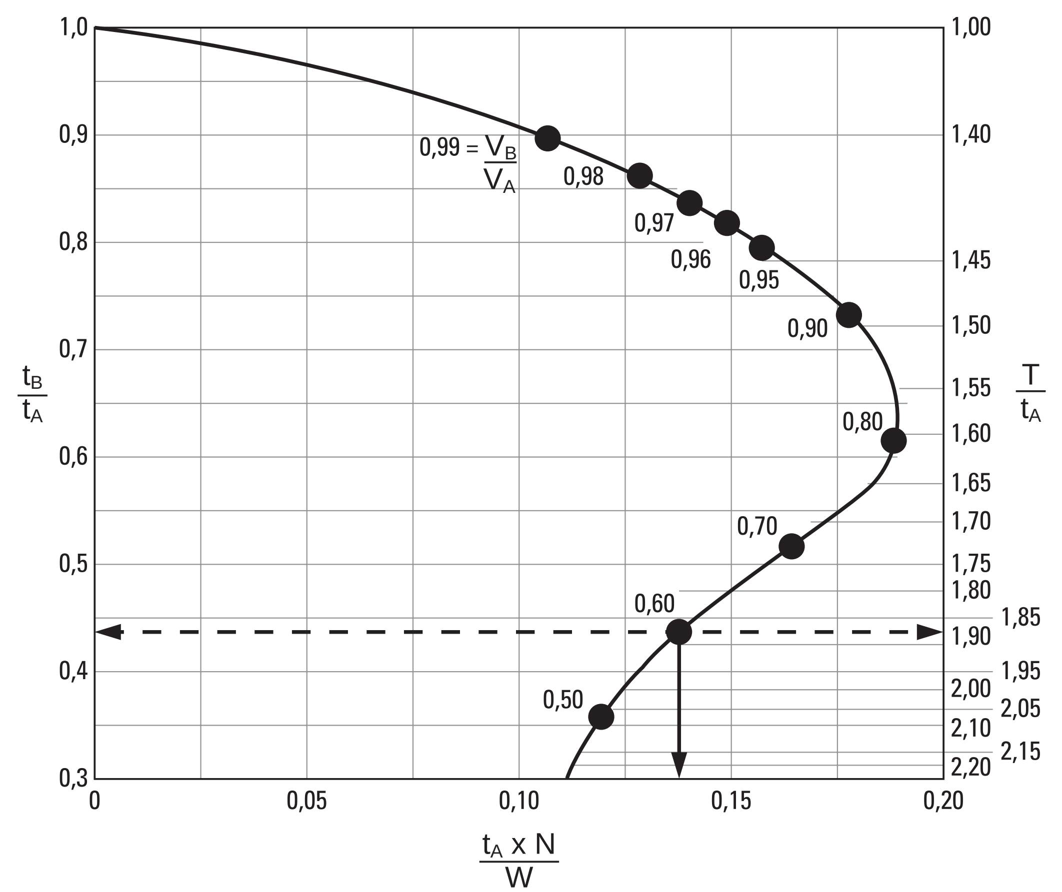


Fig. 4.08 **Ribbed plate calculator (bidirectional)**

than if the radius remained 0,5 T. However if the rib is made thinner than the wall (dotted lines in Fig. 4.07) the radius in the corners can be in proper proportion to the new rib thickness, T^1, to prevent high stress concentration and voids at the juncture, without enlarging the diameter of the enclosed circle.

Since ribbing is in such widespread use as a method to improve structure and to reduce cost and weight, simplified methods have been developed to determine the rib size and spacing necessary to provide a specified degree of rigidity. Most housings – tape cassettes, pressure containers, meter shrouds, and just plain boxes – have one functional requirement in common: the need for rigidity when a load is applied. Since rigidity is directly proportional to the moment of inertia of the housing cross section, it is physically simple (though sometimes mathematically complex) to replace a constant wall section part with a ribbed structure with the same rigidity but less weight. To simplify such analysis, the curve in Fig. 4.08 has been developed to help determine the feasibility of using a ribbed structure in a product (background, see Table 4.01).

Bidirectional ribbing

The curve in Fig. 4.08 describes the dimensional relationship between simple flat plates and cross-ribbed plates (Fig. 4.09) having identical values of moment of inertia. The base of the graph shows values from 0 to 0,2 for the product of the non-ribbed wall thickness (t_A) and the number of ribs per mm (N) divided by the width of the plate (W). The W value was taken as unity in the deveopment of the curve, thus it is always one (1).

It should be noted that the rib thickness was equated to that of the adjoining wall (t_B). However, if thinner ribs are desired to minimize sinks, their number and dimensions can easily be obtained. For example, if the ribs were 2,5 mm thick and spaced 25 mm apart, ribs which are 1,25 mm thick and spaced 12,5 mm apart would provide equivalent performance.

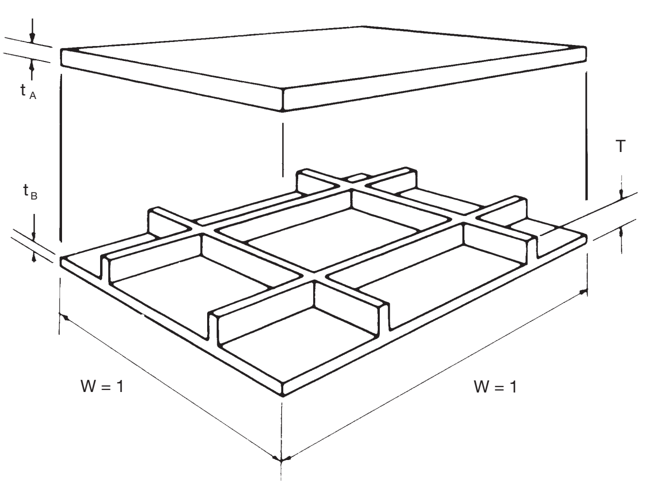


Fig. 4.09 **Equivalent flat plate and ribbed structure**

The left hand ordinate shows values from 0,3 to 1,0 for the ratio of the ribbed wall thickness (t_B) to the non-ribbed wall thickness (t_A). The right hand ordinate shows the values from 1,0 to 2,2 for the ratio of the overall thickness of the ribbed part (T) to the non-ribbed wall thickness (t_A).

Ratios of the volume of the ribbed plate (V_B) to the volume of the corresponding flat plate (V_A) are shown along the curve at spacings suitable for interpolation. For any one combination of the variables T, t_B, and N, these volume ratios will specify the minimum volume of material necessary to provide a structure equivalent to the original unribbed design, as shown in the following examples.

Example 1 – If there are no restrictions on the geometry of the new cross-ribbed wall design, the curve can be used to determine the dimension that will satisfy a required cost reduction in part weight.

Known: Present wall thickness (t_A) = 4,5 mm

Required: Material reduction equals 40%

or $\frac{V_B}{V_A} = 0{,}60$

From Fig. 4.08

$\frac{(t_A)\,(N)}{W} = 0{,}135$, or $N = \frac{0{,}135 \times 1}{4{,}5} = 0{,}03$ ribs per mm or about 3 ribs per 100 mm

$\frac{t_B}{t_A} = 0{,}44$, or $t_B = (0{,}44)\,(4{,}5) = 2{,}0$ mm

$\frac{T}{t_A} = 1{,}875$, or $T = (1{,}875)\,(4{,}5) =$ about 8,5 mm

Example 2 – If moulding flow of the resin limits the redesigned wall thickness, part geometry can be calculated as follows:

Known: Present wall thickness (t_A) = 2,5 mm

Required: Minimum wall thickness (t_B) = 1,0 mm

or $\frac{t_B}{t_A} = \frac{1{,}0}{2{,}5} = 0{,}4$

From Fig. 4.08

$\frac{T}{t_A} = 1{,}95$, or $T = (1{,}95)\,(2{,}5) = 5{,}0$ mm

$\frac{(t_A)\,(N)}{W} = 0{,}125$, or $N = \frac{0{,}125 \times 1}{2{,}5} = 0{,}05$ ribs per mm or 1 rib per 20 mm

$\frac{V_B}{V_A} = 0{,}55$

Thus, the 1,0 mm wall design has an overall height of 5,0 mm, a rib spacing of 0,05 per mm (or 1 rib every 20 mm) and provides a 45 per cent material saving.

Example 3 – If the overall wall thickness is the limitation because of internal or exterior size of the part, other dimensions can be found on the curve:

Known: Present wall thickness (t_A) = 6,5 mm

Required: Maximum height of ribbed wall (T) = 10,8 mm

$$\text{or} \quad \frac{T}{t_A} = \frac{10{,}8}{6{,}5} = 1{,}66$$

From Fig. 4.08

$$\frac{(t_A)(N)}{W} = 0{,}175, \text{ or } N = \frac{0{,}175 \times 1}{6{,}5} = 0{,}027 \text{ ribs per mm}$$
or 1 rib per 37 mm

$$\frac{t_B}{t_A} = 0{,}56, \text{ or } t_B = (0{,}56)(6{,}5) = 3{,}65 \text{ mm}$$

$$\frac{V_B}{V_A} = 0{,}76$$

The ribbed design provides a material reduction of 24 per cent, will use 0,027 ribs per mm (1 rib every 37 mm) and will have a wall thickness of 3,65 mm. If thinner ribs are desired for functional or appearance reasons, the same structure can be obtained by holding the product of the number of ribs and the rib thickness constant. In this example, if the rib wall thickness were cut in half to 1,8 mm, the number of ribs should be increased from 1 every 37 mm to 2 every 37 mm.

Example 4 – If the number of ribs per cm is limited because of possible interference with internal components of the product, or by the need to match rib spacing with an adjoining structure or decorative elements, the designer can specify the number of ribs and then determine the other dimensions which will provide a minimum volume.

Known: Present wall thickness (t_A) = 4,0 mm

Required: Ribs per mm (N) = 0,04 ribs per mm or 4 ribs per 100 mm

Therefore, for a base (W) of unity:

$$\frac{(t_A)(N)}{W} = \frac{(4{,}0)(0{,}04)}{1} = 0{,}16$$

From Fig. 4.08

$$\frac{t_B}{t_A} = 0{,}5, \text{ or } t_B = 0{,}5 \times 4{,}0 = 2{,}0 \text{ mm}$$

$$\frac{T}{t_A} = 1{,}75, \text{ or } T = 1{,}75 \times 4{,}0 = \text{about } 7{,}0 \text{ mm}$$

$$\frac{V_B}{V_A} = 0{,}68$$

The resulting design has an overall height of 7,0 mm, a wall thickness of about 2,0 mm and a material saving of 32 per cent. (An alternate solution obtained with a V_B/V_A value of 0,90 provides a material saving of only 10 per cent. The choice depends on the suitability of wall thicknesses and overall height.)

Unidirectional Ribbing

Curves have been developed which compare by means of dimensionless ratios, the geometry of flat plates and unidirectional ribbed structures of equal rigidity. The thickness of the unribbed wall, typically, would be based on the calculations an engineer might make in substituting plastic for metal in a structure that must withstand a specified loading. When the wide, rectangular cross section of that wall is analyzed, its width is divided into smaller equal sections and the moment of inertia for a single section is calculated and compared with that of its ribbed equivalent. The sum of the small section moments of inertia is equal to that of the original section.

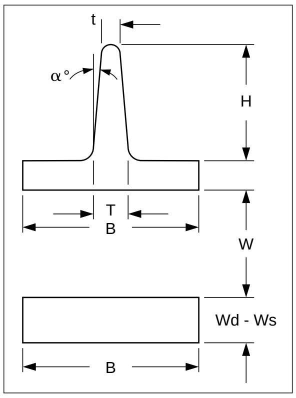
The nomenclature for the cross-section are shown below:

$$t = T - 2H \tan \alpha$$

$$A \text{ (area)} = BW + \frac{H(T + t)}{2}$$

W_d = Thickness for deflection

W_S = Thickness for stress



To define one of the smaller sections of the whole structure, the equivalent width BEQ is used.

$$BEQ = \frac{\text{total width of section}}{\text{number of ribs}} = \frac{B}{N}$$

Based on the moment of inertia equations for these sections, the thickness ratios were determined and plotted. These calculations were based on a rib thickness equal to 60 per cent of the wall thickness. The curves in Figures 4.10 and 4.11 are given in terms of the wall thickness ratio for deflection (W_d / W) or thickness ratio for stress (W_S / W).

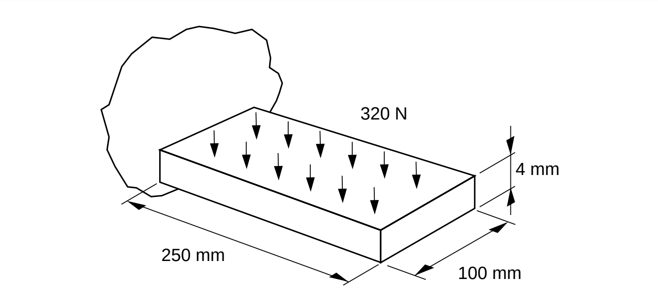
The abscissae are expressed in terms of the ratio of rib height to wall thickness (H/W). The following problems and their step by step solutions illustrate how use of the curves can simplify deflection and stress calculations.

Problem 1

A 4 mm thick copper plate, fixed at one end and subject to a uniform loading of 320 N, is to be replaced by a plate moulded in DELRIN® acetal resin. Determine the equivalent ribbed section for the new plate; dimensions, see sketch next page.

Flex modulus for copper:
E_C = 105 000 MPa

Flex modulus for DELRIN® acetal resin
E_D = 3000 MPa



The wall thickness for a plate in DELRIN® acetal resin with equivalent stiffness is calculated by equating the product of the modulus and moment of inertia of the two materials.

$E_C \times W_C^3 = E_D \times W_d^3$; or: $105000 \times 4^3 = 3000 \times W_d^3$

Thus: W_d = 13 mm.

Since a wall thickness of 13 mm is not ordinarily considered practical for plastic structures, primarily because of processing difficulties, a ribbed section is recommended. Therefore, assume a more reasonable wall of 3 mm, and compute for a plate with nine equally spaced ribs, rib height, deflection and stress.

$$\frac{W_D}{W} = \frac{13}{3} = 4{,}33$$

$$BEQ = \frac{B}{N} = \frac{100}{9} = 11{,}1 \quad \frac{BEQ}{W} = \frac{11{,}1}{3} = 3{,}7$$

From the deflection graph (Fig. 4.10) we obtain:

$$\frac{H}{W} = 5{,}7 \qquad H = 5{,}7 \times 3 = 17{,}1 \text{ mm}$$

From the stress graph (Fig. 4.11) for $\frac{H}{W} = 5{,}7$ and

$\frac{BEQ}{W} = 3{,}7$ we obtain:

$$\frac{W_s}{W} = 2{,}75 \qquad W_s = 2{,}75 \times 3 \text{ mm} = 8{,}25 \text{ mm}$$

Determine the moment of inertia and section modulus for the ribbed area, equal to that of the solid plastic plate:

$$I = B\,\frac{W_D^3}{12} = \frac{100 \times 13^3}{12} = 18\,300 \text{ mm}^4$$

$$Z = \frac{BW_s^2}{6} = \frac{100 \times 8{,}25^2}{6} = 1130 \text{ mm}^3$$

Maximum deflection at the free end:

$$\delta \max = \frac{FL^3}{8\,EI} = \frac{320 \times 250^3}{8 \times 3000 \times 18\,300} = 11{,}4 \text{ mm}$$

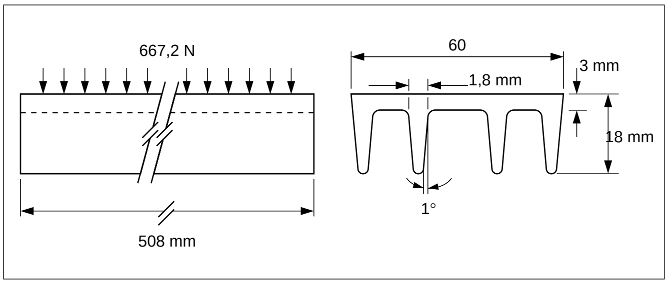
Maximum stress at the fixed end:

$$\sigma \max = \frac{FL}{2\,Z} = \frac{320 \times 250}{2 \times 1130} = 35{,}4 \text{ MPa}$$

Since DELRIN® acetal resin has a tensile strength value of 69 MPa a safety factor of 2 is obtained.

Problem 2

Determine deflection and stress for a structure as shown made of RYNITE® 530 thermoplastic polyester resin; supported at both ends.



Substitute the known data:

$$BEQ = \frac{B}{N} = \frac{60}{4} = 15 \qquad \frac{BEQ}{W} = \frac{15}{3} = 5$$

$$H = 18 - 3 = 15 \qquad \frac{H}{W} = \frac{15}{3} = 5$$

From the graphs

$$\frac{W_d}{W} = 3{,}6 \qquad W_d = 3{,}6 \times 3 = 10{,}8$$

$$\frac{W_s}{W} = 2{,}25 \qquad W_s = 2{,}25 \times 3 = 6{,}75 \text{ mm}$$

$$I = \frac{BW_d^3}{12} = \frac{60 \times 10{,}8^3}{12} = 6300 \text{ mm}^4$$

$$Z = \frac{BW_s^2}{6} = \frac{60 \times 6{,}75^2}{6} = 455 \text{ mm}^3$$

$$\delta \max = \frac{5}{384} \times \frac{FL^3}{EI} = \frac{5 \times 667{,}2 \times 508^3}{384 \times 9000 \times 6300} = 20 \text{ mm}$$

$$\sigma \max = \frac{FL}{8\,Z} = \frac{667{,}2 \times 508}{8 \times 455} = 93 \text{ MPa}$$

Since RYNITE® 530 has a tensile strength value of 158 MPa, there will be a safety factor of approximately 1,7, assuming the gate position is such, that can be counted on “in-flow” properties.

Remark: Ribs having a height exceeding 5 times their thickness and subject to higher compression stresses, should be checked on danger for buckling (instability).

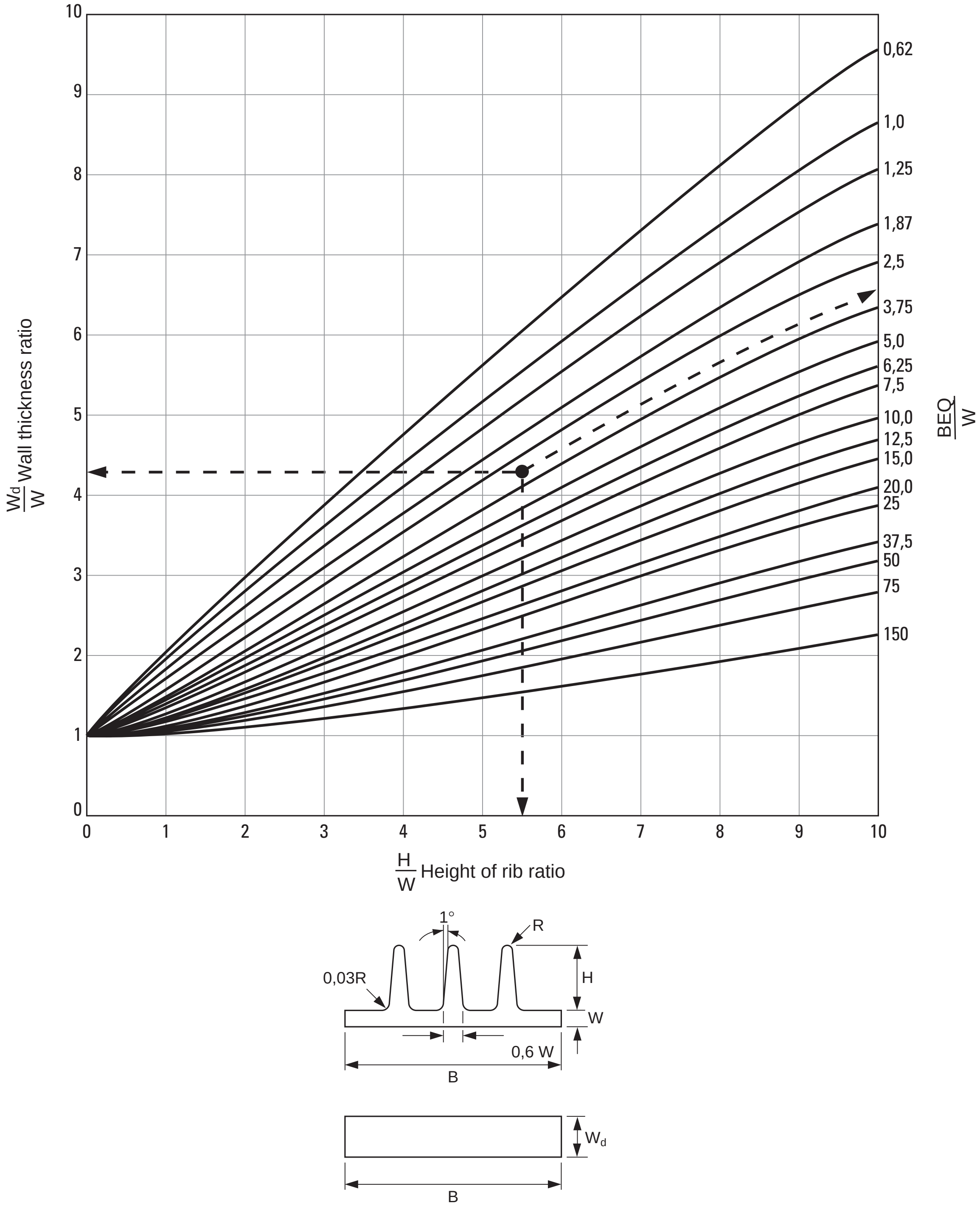


Fig. 4.10 **Deflection curves**
The computer programmed curves in this graph above, plotted for rib thicknesses equal to 60 per cent of wall thickness, are presented as an aid in calculating maximum deflection of a ribbed structure.
(For other rib thicknesses, use formulae of Tables 4.01 and 4.02).

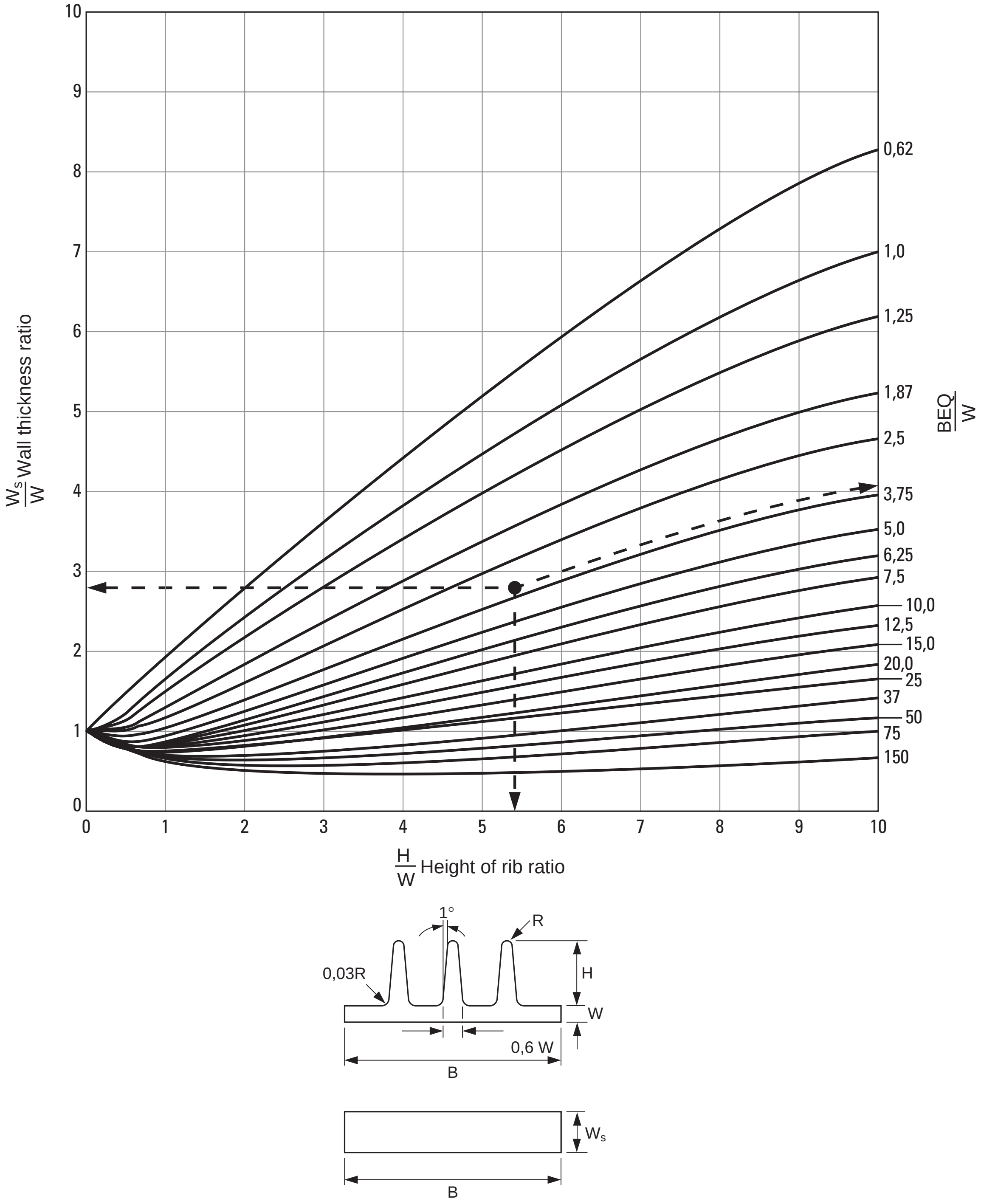
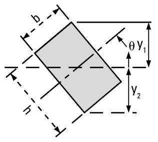
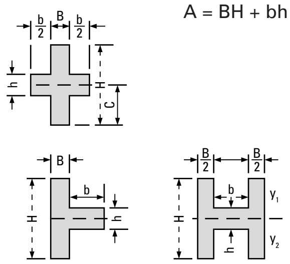
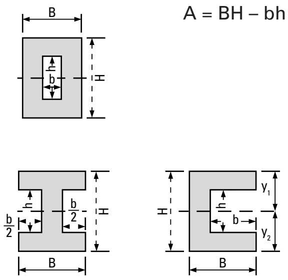
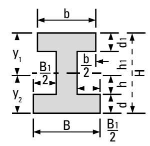
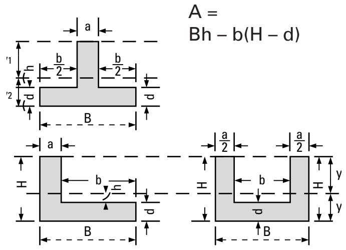
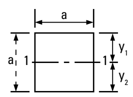
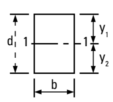
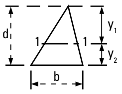
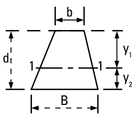
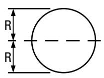
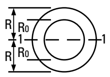
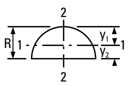
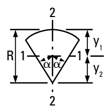
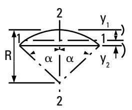
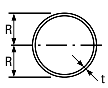
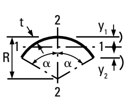


Fig. 4.11 **Stress curves**
The computer programmed curves in this graph above, plotted for rib thickness equal to 60 per cent of wall thickness, are presented as an aid in calculating the maximum stress tolerance of a ribbed structure.

Structural Design Formulae

Table 4.01 **Properties of Sections**

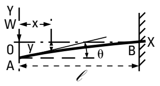
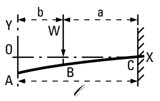
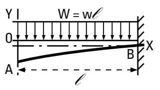
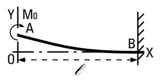
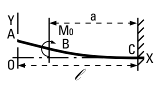
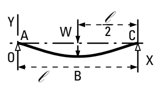
Form of section	Area A	Distance from centroid to extremities of section y_1, y_2	Moments of inertia I_1 and I_2 about principal central axis 1 and 2	Radii of gyration r_1 and r_2 about principal central axes
	A = bh	$y_1 = y_2 = \frac{h \cos \theta + b \sin \theta}{2}$	$I_1 = \frac{bh}{12}(h^2 \cos^2 \theta + b^2 \sin^2 \theta)$	$r_1 = \sqrt{\frac{(h^2 \cos^2 \theta + b^2 \sin^2 \theta)}{12}}$
	A = BH + bh	$y_1 = y_2 = \frac{H}{2}$	$I_1 = \frac{BH^2 + bh^3}{12}$	$r_1 = \sqrt{\frac{BH^3 + bh^3}{12\ (BH + bh)}}$
	A = BH – bh	$y_1 = y_2 = \frac{H}{2}$	$I_1 = \frac{BH^3 - bh^3}{12}$	$r_1 = \sqrt{\frac{BH^3 - bh^3}{12\ (BH - bh)}}$
	A = $bd_1 + Bd$ + $a(H - d - d_1)$	$y_1 = H - y_2$ $y_2 = \frac{1}{2}\,\frac{aH^2 + B_1d^2 + b_1d_1\ (2H - d_1)}{aH + B_1d + b_1d_1}$	$I_1 = \frac{1}{3}(By_2^3 - B_1h^3 + by_1^3 - b_1h_1^3)$	$r_1 = \sqrt{\frac{I}{(Bd + bd_1) + a(h + h_1)}}$
	A = $Bh - b(H - d)$	$y_1 = H - y_2$ $y_2 = \frac{aH^2 + bd^2}{2(aH + bd)}$	$I_1 = \frac{1}{3}(By_2^3 - bh^3 + ay_1^3)$	$r_1 = \sqrt{\frac{I}{Bd + a(H - d)}}$
	$A = a^2$	$y_1 = y_2 = \frac{1}{2}a$	$I_1 = I_2 = I_3 = \frac{1}{12}a^4$	$r_1 = r_2 = r_3 = 0.289a$
	A = bd	$y_1 = y_2 = \frac{1}{2}d$	$I_1 = \frac{1}{12}bd^3$	r_1 0.289d

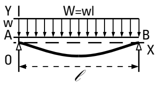
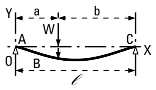
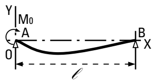
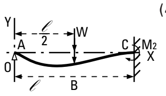
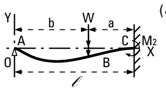
Form of section	Area A	Distance from centroid to extremities of section y_1, y_2	Moments of inertia I_1 and I_2 about principal central axis 1 and 2	Radii of gyration r_1 and r_2 about principal central axes
	$A = \frac{1}{2} bd$	$y_1 = \frac{2}{3} d$ $y_2 = \frac{1}{3} d$	$I_1 = \frac{1}{36} bd^3$	$r_1 = 0.2358d$
	$A = \frac{1}{2}(B + b)d$	$y_1 = d\,\frac{2B + b}{3(B + b)}$ $y_2 = d\,\frac{B + 2b}{3(B + b)}$	$I_1 = \frac{d^3\,(B^2 + 4Bb + b^2)}{36(B + b)}$	r1 = $\frac{d}{6(B + b)}\sqrt{2(B^2 + 4Bb + b^2)}$
	$A = \pi R^2$	$y_1 = y_2 = R$	$I = \frac{1}{4}\pi R^4$	$r = \frac{1}{2} R$
	$A = \pi (R^2 - R_0^2)$	$y_1 = y_2 - R$	$I = \frac{1}{4}\pi (R^4 - R_0^4)$	$r = \sqrt{\frac{1}{4}(R^2 + R_0^2)}$
	$A = \frac{1}{2}\pi R^2$	$y_1 = 0.5756R$ $y_2 = 0.4244R$	$I_1 = 0.1098R^4$ $I_2 = \frac{1}{8}\pi R^4$	$r_1 = 0.2643R$ $r_2 = \frac{1}{2}R$
	$A = \alpha R^2$	$y_1 = R\left(1 - \frac{2\sin\alpha}{3\alpha}\right)$ $y_2 = 2R\,\frac{\sin\alpha}{3\alpha}$	$I_1 = \frac{1}{4}R^4\left[\alpha + \sin\alpha\cos\alpha - \frac{16\sin^2\alpha}{9\alpha}\right]$ $I_2 = \frac{1}{4}R^4[\alpha - \sin\alpha\cos\alpha]$	$r_1 =$ $\frac{1}{2}R\sqrt{1 + \frac{\sin\alpha\cos\alpha}{\alpha} - \frac{16\sin^2\alpha}{9\alpha^2}}$ $r_2 = \frac{1}{2}R\sqrt{1 - \frac{\sin\alpha\cos\alpha}{\alpha}}$
(1) 	$A = \frac{1}{2}R^2(2\alpha - \sin 2\alpha)$	$y_1 = R\left(1 - \frac{4\sin^3\alpha}{6\alpha - 3\sin 2\alpha}\right)$ $y_2 = R\left(\frac{4\sin^3\alpha}{6\alpha - 3\sin 2\alpha} - \cos\alpha\right)$	$I_1 = \frac{R^4}{4}\left[\alpha + \sin\alpha\cos\alpha + 2\sin^3\alpha\cos\alpha - \frac{16\sin^6\alpha}{9(\alpha - \sin\alpha\cos\alpha)}\right]$ $I_2 = \frac{R^4}{12}[3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha\cos\alpha]$	$r_2 = \frac{1}{2}R\sqrt{1 + \frac{2\sin^3\alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha} - \frac{64\sin^6\alpha}{9(2\alpha - \sin 2\alpha)^2}}$ $r_2 = \frac{1}{2}R\sqrt{1 - \frac{2\sin^3\alpha\cos\alpha}{3(\alpha - \sin\alpha\cos\alpha)}}$
(2) 	$A = 2\pi Rt$	$y_1 = y_2 = R$	$I = \pi R^3 t$	$r = 0.707R$
Thin wall (3)  (for thick wall: see "Roark & Young	$A = \alpha(2R\text{-}t)\,t$	$y_1 = R\left(\frac{1 - \sin\alpha}{\alpha}\right) + \frac{t}{2}$ $y_2 = 2R\left(\frac{\sin\alpha}{\alpha} - \cos\alpha\right) + \frac{t}{2}\cos\alpha$	$I_1 = R^3 t\left(\alpha + \sin\alpha\cos\alpha - \frac{2\sin^2\alpha}{\alpha}\right) + \frac{\alpha Rt^3}{6}$ $I_2 = R^3 t(\alpha - \sin\alpha\cos\alpha)$	$r_1 =$ $R\sqrt{\frac{\alpha + \sin\alpha\cos\alpha - 2\sin^2\alpha/\alpha}{2\alpha}}$ $r_2 = R\sqrt{\frac{\alpha - \sin\alpha\cos\alpha}{2\alpha}}$

(1) Circular sector
(2) Very thin annulus
(3) Sector of thin annulus

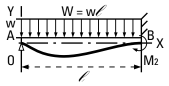
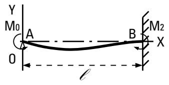
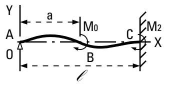
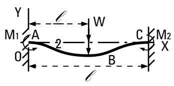
Table 4.02 **Shear, Moment, and Deflection Formulae for Beams; Reaction Formulae for Rigid Frames**

Notation: W = load (N); w = unit load (N/linear mm); M is positive when clockwise; V is positive when upward; y is positive when upward. Constraining moments, applied couples, loads, and reactions are positive when acting as shown. All forces are in N, all moments in N · mm; all deflections and dimensions in mm. θ is in radians, I = moment of inertia of beam cross section (mm^4).

Loading, support and reference number	Reactions R_1 and R_2, vertical shear V	Bending moment M and maximum bending moment	Deflection y, maximum deflection, and end slope θ
Cantilever end load 	$R_2 = +W$ $V = -W$	$M = -Wx$ Max $M = -W\ell$ at B	$y = \frac{1}{6}\frac{W}{EI}(x^3 - 3\ell^2 x + 2\ell^3)$ Max $y = -\frac{1}{3}\frac{W\ell^3}{EI}$ at A $\theta = +\frac{1}{2}\frac{W\ell^2}{EI}$ at A
Cantilever, intermediate load 	$R_2 = +W$ (A to B) $V = 0$ (B to C) $V = -W$	(A to B) $M = 0$ (B to C) $M = -W(x - b)$ Max $M = -Wa$ at C	(A to B) $y = -\frac{1}{6}\frac{W}{EI}(-a^3 + 3a^2\ell - 3a^2x)$ (B to C) $y = -\frac{1}{6}\frac{W}{EI}[(x-b)^3 - 3a^2(x-b) + 2a^3]$ Max $y = -\frac{1}{6}\frac{W}{EI}(3a^2\ell - a^3)$ $\theta = +\frac{1}{2}\frac{Wa^2}{EI}$ (A to B)
Cantilever, uniform load 	$R_2 = +W$ $V = -\frac{W}{\ell}x$	$M = -\frac{1}{2}\frac{W}{\ell}x^2$ Max $M = -\frac{1}{2}W\ell$ at B	$y = -\frac{1}{24}\frac{W}{EI\ell}(x^4 - 4\ell^3 x + 3\ell^4)$ Max $y = -\frac{1}{8}\frac{W\ell^3}{EI}$ $\theta = +\frac{1}{6}\frac{W\ell^2}{EI}$ at A
Cantilever, end couple 	$R_2 = 0$ $V = 0$	$M = M_0$ Max $M = M_0$ (A to B)	$y = \frac{1}{2}\frac{M_0}{EI}(\ell^2 - 2\ell x + x^2)$ Max $y = +\frac{1}{2}\frac{M_0\ell^2}{EI}$ at A $\theta = -\frac{M_0\ell}{EI}$ at A
Cantilever, intermediate couple 	$R_2 = 0$ $V = 0$	(A to B) $M = 0$ (B to C) $M = M_0$ Max $M = M_0$ (B to C)	(A to B) $y = \frac{M_0 a}{EI}\left(\ell - \frac{1}{2}a - x\right)$ (B to C) $y = \frac{1}{2}\frac{M_0}{EI}[(x - \ell + a)^2 - 2a(x - \ell + a) + a^2]$ Max $y = \frac{M_0 a}{EI}\left(\ell - \frac{1}{2}a\right)$ at A $\theta = -\frac{M_0 a}{EI}$ (A to B)
End supports, center load 	$R_1 = +\frac{1}{2}W$ $R_2 = +\frac{1}{2}W$ (A to B) $V = +\frac{1}{2}W$ (B to C) $V = -\frac{1}{2}W$	(A to B) $M = +\frac{1}{2}Wx$ (B to C) $M = +\frac{1}{2}W(\ell - x)$ Max $M = +\frac{1}{4}W\ell$ at B	(A to B) $y = -\frac{1}{48}\frac{W}{EI}(3\ell^2 x - 4x^3)$ Max $y = -\frac{1}{48}\frac{W\ell^3}{EI}$ at B $\theta = -\frac{1}{16}\frac{W\ell^2}{EI}$ at A, $\theta = +\frac{1}{16}\frac{W\ell^2}{EI}$ at C

Loading, support and reference number	Reactions R_1 and R_2, vertical shear V	Bending moment M and maximum bending moment	Deflection y, maximum deflection, and end slope θ
End supports, uniform load 	$R_1 = +\frac{1}{2}W$ $R_2 = +\frac{1}{2}W$ $V = \frac{1}{2}W\left(1 - \frac{2x}{\ell}\right)$	$M = \frac{1}{2}W\left(x - \frac{x^2}{\ell}\right)$ Max $M = +\frac{1}{8}W\ell$ at $x = \frac{1}{2}\ell$	$y = -\frac{1}{24}\frac{Wx}{EI\ell}(\ell^3 - 2\ell x^2 + x^3)$ Max $y = -\frac{5}{384}\frac{W\ell^3}{EI}$ at $x = \frac{1}{2}\ell$ $\theta = -\frac{1}{24}\frac{W\ell^2}{EI}$ at A; $\theta = +\frac{1}{24}\frac{W\ell^2}{EI}$ at B
End supports, intermediate load 	$R_1 = +W\frac{b}{\ell}$ $R_2 = +W\frac{a}{\ell}$ (A to B) $V = +W\frac{b}{\ell}$ (B to C) $V = -W\frac{a}{\ell}$	(A to B) $M = +W\frac{b}{\ell}x$ (B to C) $M = +W\frac{a}{\ell}(\ell - x)$ Max $M = +W\frac{ab}{\ell}$ at B	(A to B) $y = -\frac{Wbx}{6EI\ell}[2\ell(\ell - x) - b^2 - (\ell - x)^2]$ (B to C) $y = -\frac{Wa(\ell - x)}{6EI\ell}[2\ell b - b^2 - (\ell - x)^2]$ Max $y = -\frac{Wab}{27EI\ell}(a + 2b)\sqrt{3a(a + 2b)}$ at $x = \sqrt{\frac{1}{3}a(a + 2b)}$ when $a > b$ $\theta = -\frac{1}{6}\frac{W}{EI}\left(b\ell - \frac{b^3}{\ell}\right)$ at A; $\theta = +\frac{1}{6}\frac{W}{EI}\left(2b\ell + \frac{b^3}{\ell} - 3b^2\right)$ at C;
End supports, end couple 	$R_1 = -\frac{M_0}{\ell}$ $R_1 = +\frac{M_0}{\ell}$ $V = R_1$	$M = M_0 + R_1 x$ Max $M = M_0$ at A	$y = -\frac{1}{6}\frac{M_0}{EI}\left(3x^2 - \frac{x^3}{\ell} - 2\ell x\right)$ Max $y = 0.0642\frac{M_0\ell^2}{EI}$ at $x = 0.422\ell$ $\theta = -\frac{1}{3}\frac{M_0\ell}{EI}$ at A; $\theta = +\frac{1}{6}\frac{M_0\ell}{EI}$ at B
One end fixed, one end supported. Center load (4) 	$R_1 = \frac{5}{16}W$ $R_2 = \frac{11}{16}W$ $M_2 = \frac{3}{16}W\ell$ (A to B) $V = +\frac{5}{16}W$ (B to C) $V = -\frac{11}{16}W$	(A to B) $M = \frac{5}{16}Wx$ (B to C) $M = W\left(\frac{1}{2}\ell - \frac{11}{16}x\right)$ Max $+M = \frac{5}{32}W\ell$ at B Max $-M = -\frac{3}{16}W\ell$ at C	(A to B) $y = \frac{1}{96}\frac{W}{EI}(5x^3 - 3\ell^2 x)$ (B to C) $y = \frac{1}{96}\frac{W}{EI}\left[5x^3 - 16\left(x - \frac{\ell}{2}\right)^3 - 3\ell^2 x\right]$ Max $y = -0.00932\frac{W\ell^3}{EI}$ at $x = 0.4472\ell$ $\theta = -\frac{1}{32}\frac{W\ell^2}{EI}$ at A
One end fixed, one end supported. Intermediate load (4) 	$R_1 = \frac{1}{2}W\left(\frac{3a^2\ell - a^3}{\ell^3}\right)$ $R_2 = W - R_1$ $M_2 = \frac{1}{2}W\left(\frac{a^3 + 2a\ell^2 - 3a^2\ell}{\ell^2}\right)$ (A to B) $V = +R_1$ (B to C) $V = R_1 - W$	(A to B) $M = R_1 x$ (B to C) $M = R_1 x - W(x - \ell + a)$ Max $+ M = R_1(\ell - a)$ at B; Max. possible value $= 0.174\,W\ell$ when $a = 0.634\ell$ Max $- M = -M_2$ at C; Max. possible value $= -0.1927W\ell$ when $a = 0.4227\ell$	(A to B) $y = \frac{1}{6EI}[R_1(x^3 - 3\ell^2 x) + 3Wa^2 x]$ (B to C) $y = \frac{1}{6EI}\{R_1(x^3 - 3\ell^2 x) + W[3a^2 x - (x - b)^3]\}$ if $a < 0.586\ell$, max y is between A and B at: $x = \ell\sqrt{1 - \frac{2\ell}{3\ell - a}}$ if $a > 0.586\ell$, max is at: $x = \frac{\ell(\ell^2 + b^2)}{3\ell^2 - b^2}$ if $a > 0.586\ell$, max y is at B and $x = -0.0098\frac{W\ell^3}{EI}$, max possible deflection $\theta = \frac{1}{4}\frac{W}{EI}\left(\frac{a^3}{\ell} - a^2\right)$ at A

(4) M_2 = Constraining Moment

Loading, support and reference number	Reactions R_1 and R_2, constraining moments M_1 and M_2 and vertical shear V	Bending moment M and maximum positive and negative bending moment	Deflection y, maximum deflection, and end slope θ
One end fixed, one end supported. Uniform load. 	$R_1 = \frac{3}{8}\ W$ $R_2 = \frac{5}{8}\ W$ $M_2 = \frac{1}{8}\ W\ell$ $V = W\left(\frac{3}{8} - \frac{x}{\ell}\right)$	$M = W\left(\frac{3}{8}x - \frac{1}{2}\frac{x^2}{\ell}\right)$ Max $+M = \frac{9}{128}\ W\ell$ at $x = \frac{3}{8}\ell$ Max $-M = -\frac{1}{8}\ W\ell$ at B	$y = -\frac{1}{48}\ \frac{W}{EI\ell}\ (3\ell x^3 - 2x^4 + \ell^3 x)$ Max $y = -0.0054\ \frac{W\ell^3}{EI}$ at $x = 0.4215\ell$ $\theta = -\frac{1}{24}\ \frac{W\ell^2}{EI}$ at A
One end fixed, one end supported. End couple. 	$R_1 = -\frac{3}{2}\ \frac{M_0}{\ell}$ $R_2 = +\frac{3}{2}\ \frac{M_0}{\ell}$ $M_2 = \frac{1}{2}\ M_0$ $V = -\frac{3}{2}\ \frac{M_0}{\ell}$	$M = \frac{1}{2}\ M_0\left(2 - 3\frac{x}{\ell}\right)$ Max $+M = M_0$ at A Max $-M = \frac{1}{2}\ M_0$ at B	$y = \frac{1}{4}\ \frac{M_0}{EI}\left(2x^2 - \frac{x^3}{\ell} - x\ell\right)$ Max $y = -\frac{1}{27}\ \frac{M_0\ell^2}{EI}$ at $x = \frac{1}{3}\ \ell$ $\theta = -\frac{1}{4}\ \frac{M_0\ell}{EI}$ at A
One end fixed, one end supported. Intermediate couple. 	$R_1 = -\frac{3}{2}\ \frac{M_0}{\ell}\left(\frac{\ell^2 - a^2}{\ell^2}\right)$ $R_2 = +\frac{3}{2}\ \frac{M_0}{\ell}\left(\frac{\ell^2 - a^2}{\ell^2}\right)$ $M_2 = \frac{1}{2}\ M_0\left(1 - 3\ \frac{a2}{\ell^2}\right)$ (A to B) $V = R_1$ (B to C) $V = R_1$	(A to B) $M = R_1 x$ (B to C) $M = R_1 x + M_0$ Max $+M = M_0\left[1 - \frac{3a(\ell^2 - a^2)}{2\ell^3}\right]$ at B (to right) Max $-M = -M_2$ at C (when $a < 0.275\ell$) Max $-M = R_1 a$ at B (to left) (when $a > 0.275\ell$)	(A to B) $y = \frac{M_0}{EI}\left[\frac{\ell^2 - a^2}{4\ell^3}\ (3\ell^2 x - x^3) - (\ell - a)x\right]$ (B to C) $y = \frac{M_0}{EI}\left[\frac{\ell^2 - a^2}{4\ell^3}\ (3\ell^2 x - x^3) - \ell x + \frac{1}{2}\ (x^2 + a^2)\right]$ $\theta = \frac{M_0}{EI}\left(a - \frac{1}{4}\ \ell - \frac{3}{4}\ \frac{a^2}{\ell}\right)$ at A
Both ends fixed. Center load. 	$R_1 = \frac{1}{2}\ W$ $R_2 = \frac{1}{2}\ W$ $M_1 = \frac{1}{8}\ W\ell$ $M_2 = \frac{1}{8}\ W\ell$ (A to B) $V = +\frac{1}{2}\ W$ (B to C) $V = -\frac{1}{2}\ W$	(A to B) $M = \frac{1}{8}\ W(4x - \ell)$ (B to C) $M = \frac{1}{8}\ W(3\ell - 4x)$ Max $+M = \frac{1}{8}\ W\ell$ at B Max $-M = -\frac{1}{8}\ W\ell$ at A and C	(A to B) $y = -\frac{1}{48}\ \frac{W}{EI}(3\ell x^2 - 4x^3)$ Max $y = -\frac{1}{192}\ \frac{W\ell^3}{EI}$ at B

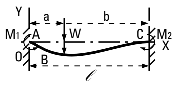
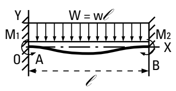
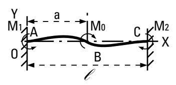
Loading, support and reference number	Reactions R_1 and R_2, constraining moments M_1 and M_2 and vertical shear V	Bending moment M and maximum positive and negative bending moment	Deflection y, maximum deflection, and end slope θ
Both ends fixed. Intermediate couple. 	$R_1 = \frac{Wb^2}{\ell^3}(3a + b)$ $R_2 = \frac{Wa^2}{\ell^3}(3b + a)$ $M_1 = W\frac{ab^2}{\ell^2}$ $M_2 = W\frac{a^2b}{\ell^2}$ (A to B) $V = R_1$ (B to C) $V = R_1 - W$	(A to B) $M = -W\frac{ab^2}{\ell^2} + R_1x$ (B to C) $M = -W\frac{ab^2}{\ell^2} + R_1x - W(x - a)$ Max + $M = -W\frac{ab^2}{\ell^2} + R_1$ at B; max possible value = $\frac{1}{8}W\ell$ when $a = \frac{1}{2}\ell$ Max − $M = -M_1$ when $a < b$; max possible value = −0.1481 $W\ell$ when $a = \frac{1}{3}\ell$ Max − $M = -M_2$ when $a > b$; max possible value = −0.1481 $W\ell$ when $a = \frac{2}{3}\ell$	(A to B) $y = \frac{1}{6}\frac{Wb^2x^2}{EI\ell^3}(3ax + bx - 3a\ell)$ (B to C) $y = \frac{1}{6}\frac{Wa^2(\ell - x)^2}{EI\ell^3}[(3b + a)(\ell - x) - 3b\ell]$ Max $y = -\frac{2}{3}\frac{W}{EI}\frac{a^3b^2}{(3a + b)^2}$ at $x = \frac{2a\ell}{3a + b}$ if $a > b$ Max $y = -\frac{2}{3}\frac{W}{EI}\frac{a^2b^3}{(3b + a)^2}$ at $x = \ell - \frac{2b\ell}{3b + a}$ if $a < b$
Both ends fixed. Uniform load. 	$R_1 = \frac{1}{2}W$ $R_2 = \frac{1}{2}W$ $M_1 = \frac{1}{12}W\ell$ $M_2 = \frac{1}{12}W\ell$ $V = \frac{1}{2}W\left(\frac{1 - 2x}{\ell}\right)$	$M = \frac{1}{2}W\left(x - \frac{x^2}{\ell} - \frac{1}{6}\ell\right)$ Max + $M = \frac{1}{24}W\ell$ at $x = \frac{1}{2}\ell$ Max − $M = -\frac{1}{12}W\ell$ at A and B	$y = \frac{1}{24}\frac{Wx^2}{EI\ell}(2\ell x - \ell^2 - x^2)$ Max $y = -\frac{1}{384}\frac{W\ell^3}{EI}$ at $x = \frac{1}{2}\ell$
Both ends fixed. Intermediate couple. 	$R_1 = -6\frac{M_0}{\ell^3}(a\ell - a^2)$ $R_2 = 6\frac{M_0}{\ell^3}(a\ell - a^2)$ $M_1 = -\frac{M_0}{\ell^2}(4\ell a - 3a^2 - \ell^2)$ $M_2 = \frac{M_0}{\ell^2}(2\ell a - 3a^2)$ $V = R_1$	(A to B) $M = -M_1 + R_1x$ (B to C) $M = -M_1 + R_1x + M_0$ Max + $M = M_0\left(4\frac{a}{\ell} - 9\frac{a^2}{\ell^2} + 6\frac{a^3}{\ell^3} - 1\right)$ just right of B Max + $M = M_0\left(4\frac{a}{\ell} - 9\frac{a^2}{\ell^2} + 6\frac{a^3}{\ell^3}\right)$ just left of B	(A to B) $y = -\frac{1}{6EI}(3M_1x^2 - R_1x^3)$ (B to C) $y = \frac{1}{6EI}[(M_0 - M_1)(3x^2 - 6\ell x + 3\ell^2)$ $- R_1(3\ell^2x - x^3 - 2\ell^3)]$ Max − y at $x = \frac{2M_1}{R_1}$ if $a > \frac{1}{3}\ell$ Max − y at $x = \ell - \frac{2M_2}{R_2}$ if $a < \frac{2}{3}\ell$

Table 4.03 **Formulae for Torsional Deformation and Stress**

General formulae: $\theta = \frac{M_T L}{KG}$, $\tau = \frac{M_T}{Q}$, where θ = angle of twist (rad); M_T = twisting moment (N · mm);

ℓ = length (mm); τ = unit shear stress (MPa); G = shear modulus (MPa); K (mm^4) is a function of the cross section.

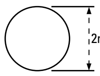
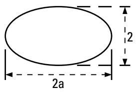
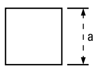
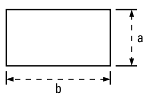
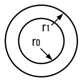
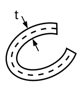
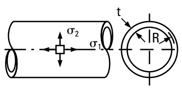
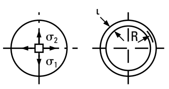
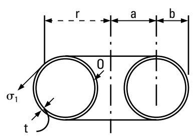
Form an dimensions of cross sections	Formula for K in $\theta = \frac{M_T \ell}{KG}$	Formula for shear stress
Solid circular section 	$K = \frac{1}{2}\pi r^4$	Max $\tau = \frac{2M_T}{\pi r^3}$ at boundary
Solid elliptical section 	$K = \frac{\pi a^3 b^3}{a^2 + b^2}$	Max $\tau = \frac{2M_T}{\pi ab^2}$ at ends for minor axis
Solid square section 	$K = 0.1406a^4$	Max $\tau = \frac{M_T}{0.208a^3}$ at mid-point of each side
Solid rectangular section  (b ≥ a)	$K = a^3 b\left[\frac{1}{3} - 0.28\ \frac{a}{b}\left(1 - \frac{a^4}{12b^4}\right)\right]$	Max $\tau = \frac{M_T(1.8a + 3.0b)}{a^2 b^2}$ at mid-point of longer edges
Hollow concentric circular section 	$K = \frac{1}{2}\ \pi\ (r_1^4 - r_0^4)$	Max $\tau = \frac{2M_T r_1}{\pi\ (r_1^4 - r_0^4)}$ at outer boundary
Any thin open tube of uniform thickness U = length of median line, shown dotted 	$K = \frac{1}{2}\ Ut^3$	Max $\tau = \frac{M_T(3U + 1.8t)}{U^2 t^2}$, along both edges remote from ends (this assumes t small compared with least radius of curvatore of median line)

Table 4.04 **Formulae for Stresses and Deformations in Pressure Vessels**

Notation for thin vessels: p = unit pressure (MPa); σ_1 = meridional membrane stress, positive when tensile (MPa); σ_2 = hoop membrane stress, positive when tensile (MPa); τ_s = shear stress (MPa); R = mean radius of circumference (mm); t = wall thickness (mm); E = modulus of elasticity (MPa); v = Poisson's ratio.

Notation for thick vessels: σ_1 = meridional wall stress, positive when acting as shown (MPa); σ_2 = hoop wall stress, positive when acting as shown (MPa); σ_3 = radial wall stress, positive when acting as shown (MPa); a = inner radius of vessel (mm); b = outer radius of vessel (mm); r = radius from axis to point where stress is to be found (mm); Δa = change in inner radius due to pressure, positive when representing an increase (mm); Δb = change in outer radius due to pressure, positive when representing an increase (mm). Other notation same as that used for thin vessels.

Form of vessel	Manner of loding	Formulas
Thin vessels – membrane stresses σ_1 (meridional) and σ_2 (hoop)		
Cylindrical 	Uniform internal (or external) pressure p, MPa	$\sigma_1 = \frac{pR}{2t}$ $\sigma_2 = \frac{pR}{t}$ Radial displacement = $\frac{R}{E}$ $(\sigma_2 - v\sigma_1)$. External collapsing pressure $p' = \frac{t}{R}\left(\frac{\sigma_y}{1 + 4\frac{\sigma_y}{E}\left(\frac{R}{t}\right)^2}\right)$ Internal bursting pressure $p_u = 2\,\sigma_u\,\frac{t}{R}$ Here σ_u = ultimate tensile strength, where σ_y = compressive yield point of material. This formula is for nonelastic failure, and holds only when $\frac{p'R}{t}$ > proportional limit.
Spherical 	Uniform internal (or external) pressure p, MPa	$\sigma_1 = \sigma_2 = \frac{pR}{2t}$ Radial displacement = $\frac{\sigma_1}{E}$ $(1 - v)\,R$
Thick vessels – membrane stresses σ_1 (meridional), σ_2 (hoop) and σ_3 (radial)		
Torus 	Complete torus under uniform internal pressure p, MPa	$\sigma_1 = \frac{pb}{t}\left(\frac{1 + a}{2r}\right)$ Max $\sigma_1 = \frac{pb}{t}\left(\frac{2a - b}{2a - 2b}\right)$ at 0 $\sigma_2 = \frac{pR}{2t}$ (uniform throughout)

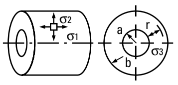
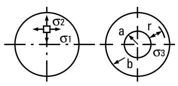
Form of vessel	Manner of loding	Formulas
Thick vessels – wall stress σ_1 (longitudinal), σ_2 (circumferential) and σ_3 (radial)		
Cilindrical 	1. Uniform internal radial pressure *p* MPa (longitudinal pressure zero or externally balanced)	$\sigma_1 = 0$ $\sigma_2 = p\,\frac{a^2(b^2+r^2)}{r^2(b^2-a^2)}$. Max $\sigma_2 = p\,\frac{b^2+a^2}{b^2-a^2}$ at inner surface $\sigma_3 = -p\,\frac{a^2(b^2-r^2)}{r^2(b^2-a^2)}$. Max $\sigma_3 = -p$ at inner surface Max $\tau = p\,\frac{b^2}{b^2-a^2}$ at inner surface $\Delta a = p\,\frac{a}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\nu\right)$; $\Delta b = p\,\frac{b}{E}\left(\frac{2a^2}{b^2-a^2}\right)$
	2. Uniform external radial pressure *p* MPa	$\sigma_1 = 0$ $\sigma_2 = -p\,\frac{a^2(b^2+r^2)}{r^2(b^2-a^2)}$. Max $\sigma_2 = -p\,\frac{2b^2}{b^2-a^2}$ at inner surface $\sigma_3 = -p\,\frac{b^2(r^2-a^2)}{r^2(b^2-a^2)}$. Max $\sigma_3 = -p$ at outer surface; Max $\tau = \frac{1}{2}$ max σ_2 at inner surface $\Delta a = -p\,\frac{a}{E}\left(\frac{2b^2}{b^2-a^2}\right)$; $\Delta b = -p\,\frac{b}{E}\left(\frac{a^2+b^2}{b^2-a^2}-\nu\right)$
	3. Uniform internal pressure *p* MPa in all directions (ends capped)	$\sigma_1 = p\,\frac{a^2}{b^2-a^2}$, σ_2 and σ_3 same as for Case 1. $\Delta a = p\,\frac{a}{E}\left[\frac{b^2+a^2}{b^2-a^2}-\nu\left(\frac{a^2}{b^2-a^2}-1\right)\right]$; $\Delta b = p\,\frac{b}{E}\left[\frac{a^2}{b^2-a^2}\,(2-\nu)\right]$
Spherical 	Uniform internal pressure *p* MPa	$\sigma_1 = \sigma_2 = p\,\frac{a^3(b^3+2r^3)}{2r^3(b^3-a^3)}$. Max σ_1 = max $\sigma_2 = p\,\frac{b^3+2a^3}{2(b^3-a^3)}$ at inner surface $\sigma_3 = -p\,\frac{a^3(b^3-r^3)}{r^3(b^3-a^3)}$. Max $\sigma_3 = -p$ at inner surface; Max $\tau = p\,\frac{3b^3}{4(b^3-a^3)}$ at inner surface $\Delta a = p\,\frac{a}{E}\left[\frac{b^3+2a^3}{2(b^3-a^3)}(1-\nu)+\nu\right]$; $\Delta b = p\,\frac{b}{E}\left[\frac{3a^3}{2(b^3-a^3)}\,(1-\nu)\right]$ Yield pressure $p_y = \frac{2\sigma_y}{3}\left(1-\frac{a^3}{b^3}\right)$
	Uniform external pressure *p* MPa	$\sigma_1 = \sigma_2 = -p\,\frac{b^3(a^3+2r^3)}{2r^3(b^3-a^3)}$. Max σ_1 = – max $\sigma_2 = -p\,\frac{3b^3}{2(b^3-a^3)}$ at inner surface $\sigma_3 = -p\,\frac{b^3(r^3-a^3)}{r^3(b^3-a^3)}$. Max $\sigma_3 = -p$ at outer surface; $\Delta a = -p\,\frac{a}{E}\left[\frac{3b^3}{2(b^3-a^3)}\,(1-\nu)\right]$; $\Delta b = -p\,\frac{b}{E}\left[\frac{a^3+2b^3}{2(b^3-a^3)}(1-\nu)-\nu\right]$

Table 4.05 **Buckling of Columns, Rings and Arches**

E = modulus of elasticity, I = moment of inertia of cross section about central axis perpendicular to plane of buckling. All dimensions are in mm, all forces in N, all angles in radians.

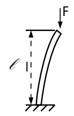
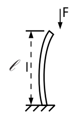
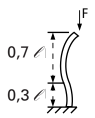
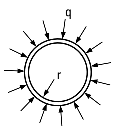
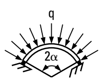
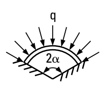
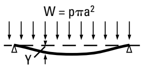
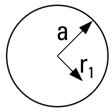
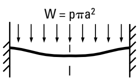
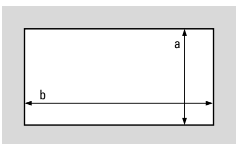
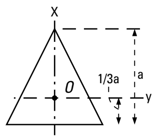
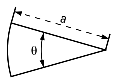
Form of bar; manner of loading and support	Formulas for critical load F_c, or critical unit load q_c
Uniform straight bar under end load One end free, other end fixed 	$F_c = \frac{\pi^2 EI}{4\ell^2}$
Uniform straight bar under end load Both ends hinged 	$F_c = \frac{\pi^2 EI}{\ell^2}$
Uniform straight bar under end load One end fixed, other end hinged and horizontally constrained over fixed end 	$F_c = \frac{\pi^2 EI}{(0.7\ell)^2}$
Uniform circular ring under uniform radial pressure *q* N·m. Mean radius of ring *r*. 	$q_c = \frac{3EI}{r^2}$
Uniform circular arch under uniform radial pressure *q*. Mean radius *r*. Ends hinged 	$q_c = \frac{EI}{r^3}\left(\frac{\pi^2}{\alpha^2} - 1\right)$
Uniform circular arch under uniform radial pressure *q*. Mean radius *r*. Ends fixed 	$q_c = \frac{EI}{r^3}(k^2 - 1)$ Where *k* depends on α and is found by trial from the equation: $k \tan \alpha \cot k\alpha = 1$ or from the following table: α = 15° 30° 45° 60° 75° 90° 120° 180° *k* = 17.2 8.62 5.80 4.37 3.50 3.00 2.36 2.00

Table 4.06 **Formulae for Flat Plates**

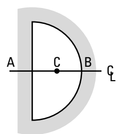
Notation: W = total applied load (N); p = unit applied load (MPa); t = thickness of plate (mm); σ = unit stress at surface of plate (MPa); y = vertical deflection of plate from original position (mm); θ = slope of plate measured from horizontal (rad); E = modulus of elasticity; ν = Poisson's ratio; r denotes the distance from the center of a circular plate. Other dimensions and corresponding symbols are indicated on figures.
Positive sign for σ indicates tension at upper surface and equal compression at lower surface; negative sign indicates reverse condition.
Positive sign for y indicates upward deflection, negative sign downward deflection. Subscripts r, t, a, and b used with σ denote respectively radial direction, tangential direction, direction of dimension a, and direction of dimension b.
All dimensions are in mm.

Manner of loading and Case No.	Formulas for stress and deflection
Edges supported Uniform load over entire surface  	Circular and solid  $(0 \le r \le a)$ (At r) $\sigma_r = -\frac{3W}{8\pi t^2}\left[(3+\nu)\left(1-\frac{r^2}{a^2}\right)\right]$ $\sigma_t = -\frac{3W}{8\pi t^2}\left[(3+\nu)-(1+3\nu)\frac{r^2}{a^2}\right]$ (At center) Max $\sigma_r = \sigma_t = -\frac{3W}{8\pi t^2}(3+\nu)$ Max $y = -\frac{3W(1-\nu)(5+\nu)a^2}{16\pi E t^3}$ (At edge) $\theta = \frac{3W(1-\nu)a}{2\pi E t^3}$
Edges fixed Uniform load over entire surface  	(At r) $\sigma_r = \frac{3W}{8\pi t^2}\left[(3+\nu)\frac{r^2}{a^2}-(1+\nu)\right]$ $\sigma_t = \frac{3W}{8\pi t^2}\left[(3+\nu)\frac{r^2}{a^2}-(1+\nu)\right]$ (At edge) Max $\sigma_r = \frac{3W}{4\pi t^2}$; $\sigma_t = \nu\frac{3W}{4\pi t^2}$ (At center) $\sigma_r = \sigma_r = -\frac{3W(1+\nu)}{8\pi t^2}$ Max $y = -\frac{3W(1-\nu^2)a^2}{16\pi E t^3}$
Uniform load over entire surface  (b ≥ a)	**Edges supported:** At centre of plate: $\sigma = \frac{0.75\,a^2\,p}{t^2\left(1+1.61\frac{a^3}{b^3}\right)}$ At centre of edge: $\sigma = 0$ Max y: $\frac{0.142\,a^4\,p}{Et^3\left(1+2.21\frac{a^3}{b^3}\right)}$ **Edges clamped:** At centre of plate: $\sigma = \frac{0.167\,a^2\,p}{t^2}$ (a=b) At centre of edge: $\sigma = \frac{0.50\,a^2\,p}{t^2\left(1+0.623\frac{a^6}{b^6}\right)}$ Max y: $\frac{0.0284\,a^4\,p}{Et^3\left(1+1.056\frac{a^5}{b^5}\right)}$

<table>
<tr><th>Manner of loading and Case No.</th><th>Formulas for stress and deflection</th></tr>
<tr><td>Edges supported
Distributed load
of intensity p over
entire surface

</td><td>Equilateral triangle, solid

$\text{Max } \sigma_x = 0.1488 \dfrac{pa^2}{t^2}$ at $y = 0$, $x = -0.062a$
$\text{Max } \sigma_y = 0.1554 \dfrac{pa^2}{t^2}$ at $y = 0$, $x = 0.129a$ (values for $\nu = 0.3$)
$\text{Max } y = \dfrac{pa^4 (1 - \nu^2)}{81Et^3}$ at centre O.</td></tr>
<tr><td>Edges supported
Distributed load
of intensity p over
entire surface

</td><td>Circular sector, solid

$\text{Max } \sigma_r = ß \dfrac{pa^2}{t^2}$ $\text{Max } \sigma_1 = ß_1 \dfrac{pa^2}{t^2}$ $\text{Max } y = \alpha \dfrac{pa^4}{Et^2}$

(values for $v = 0.3$)
<table>
<tr><th>θ</th><th>45°</th><th>60°</th><th>90°</th><th>180°</th></tr>
<tr><td>ß</td><td>0.102</td><td>0.147</td><td>0.240</td><td>0.522</td></tr>
<tr><td>$ß_1$</td><td>0.114</td><td>0.155</td><td>0.216</td><td>0.312</td></tr>
<tr><td>α</td><td>0.0054</td><td>0.0105</td><td>0.0250</td><td>0.0870</td></tr>
</table></td></tr>
<tr><td>Solid semicircular
plate, uniform load p,
all edges fixed

</td><td>$\text{Max } \sigma = \sigma_r$ in A $= \dfrac{0.42pa^2}{t^2}$
σ_r in B $= \dfrac{0.36pa^2}{t^2}$
$\text{Max } \sigma_t = \dfrac{0.21pa^2}{t^2}$ at C</td></tr>
</table>